Automated reasoning and Constraint Progamming

A brief introduction

Dimitri Justeau-Allaire

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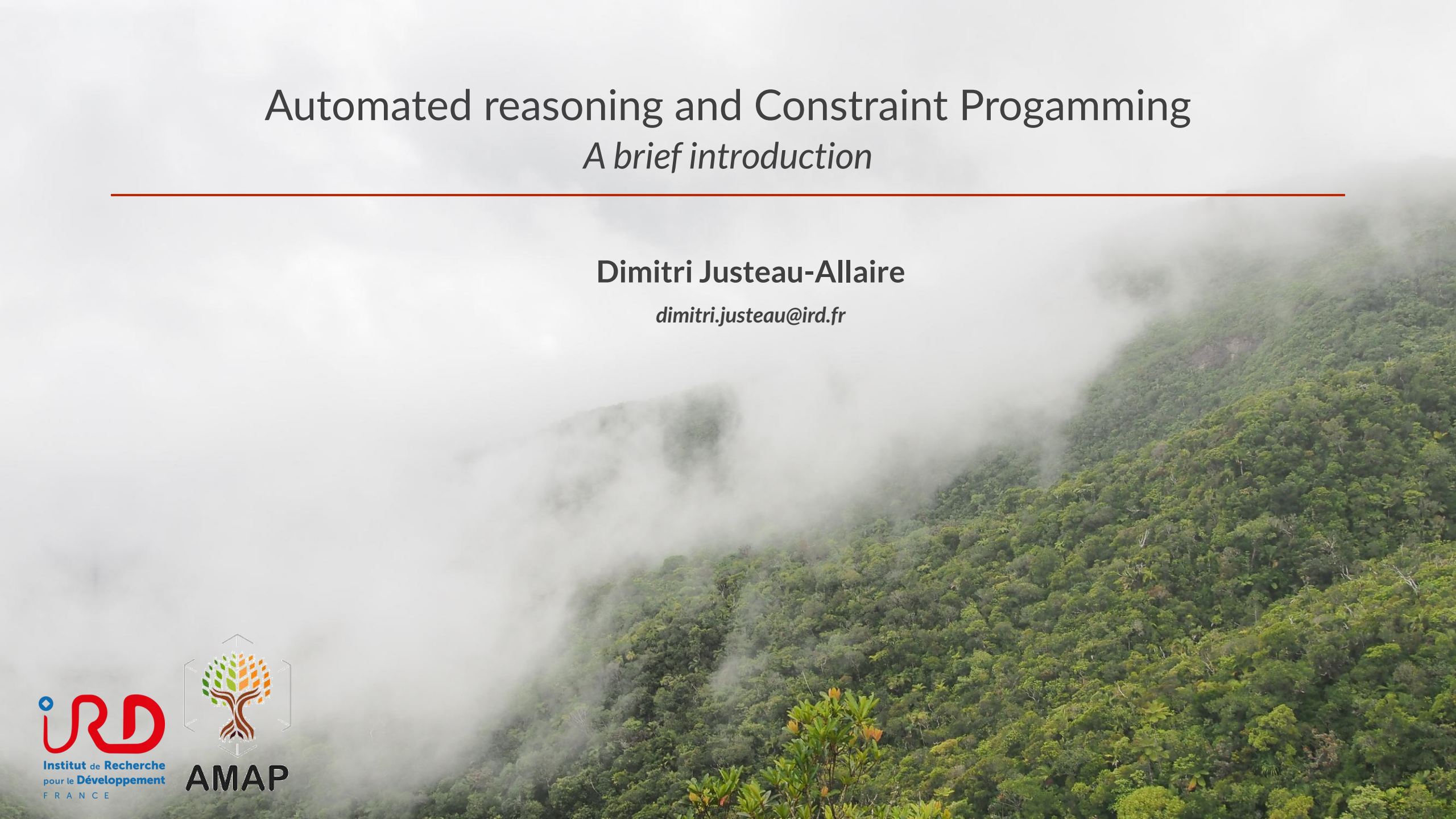

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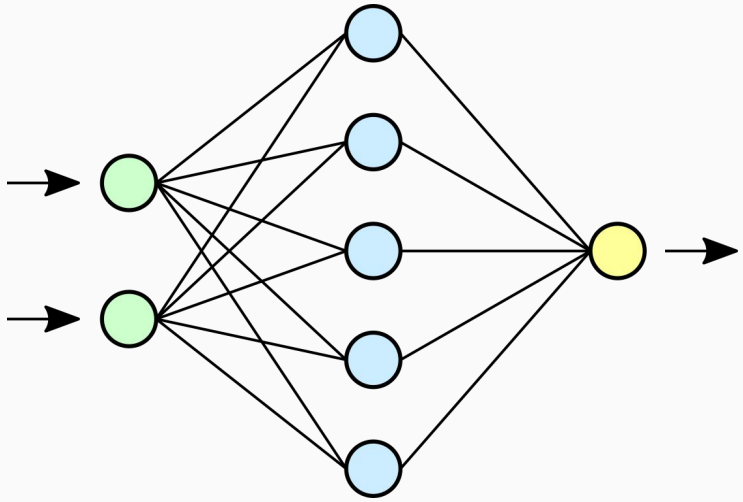
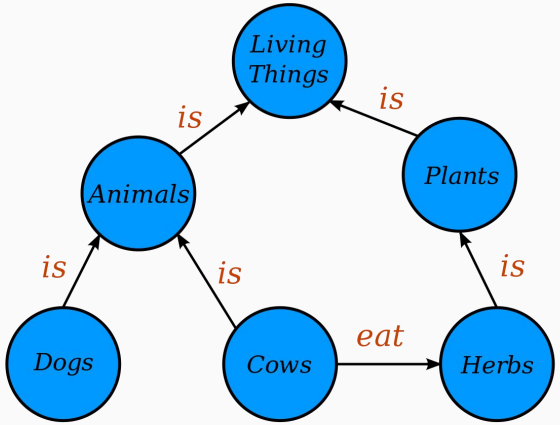
Three main «domains» of artificial intelligence

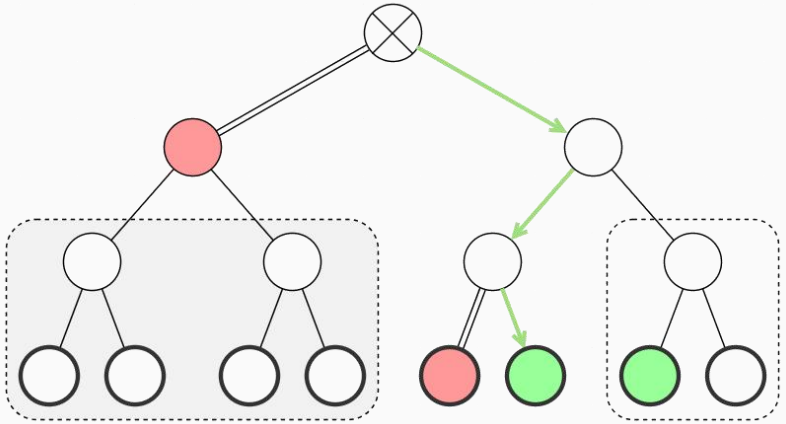
Automated learning



Knowledge representation



Automated reasoning



Three main «domains» of artificial intelligence

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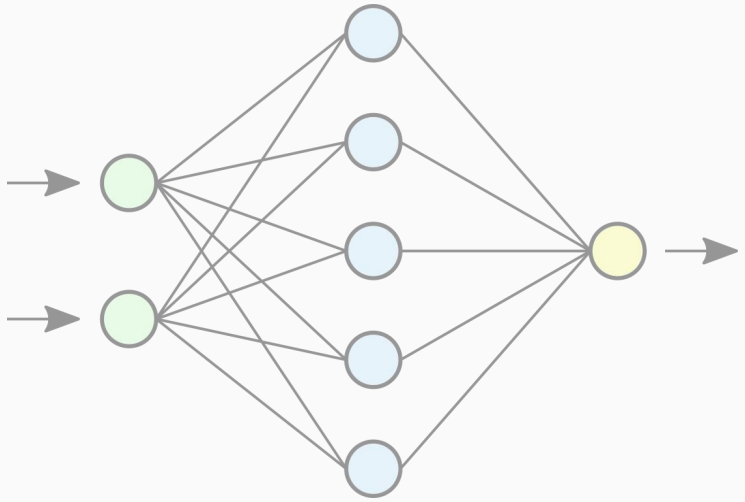

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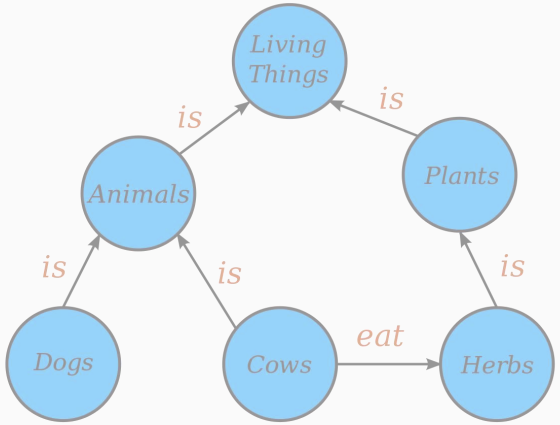

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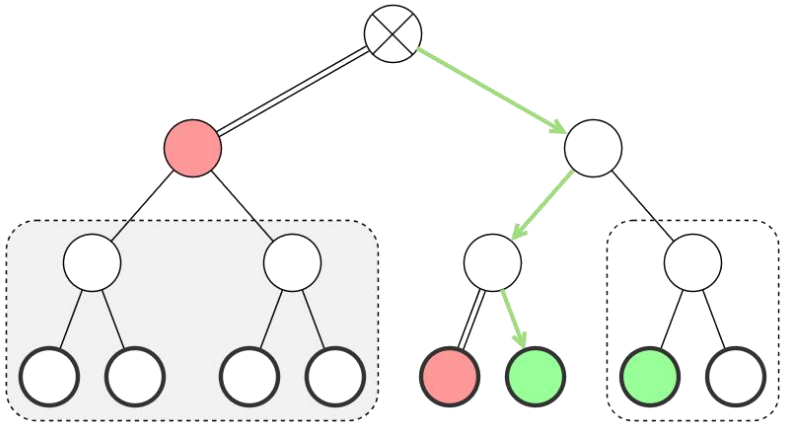

e.g. SAT, MILP, CP, expert systems, inference

Automated learning

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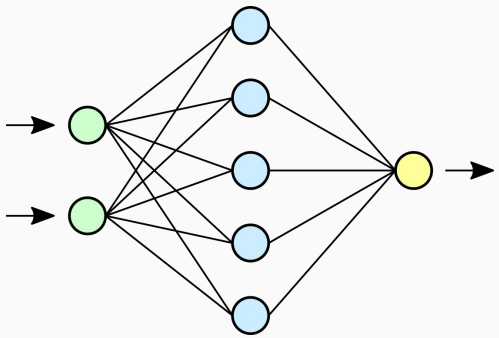
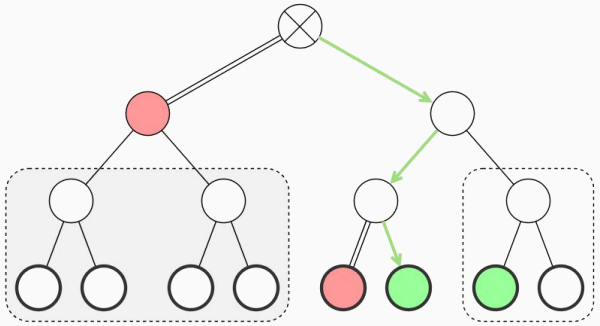




Automated learning

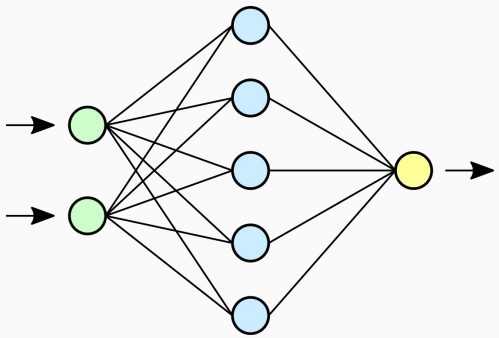
- Based on **data**

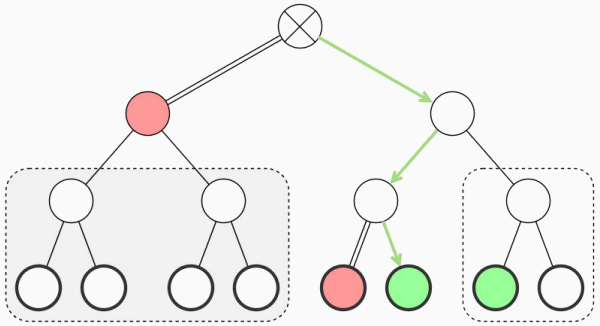
Automated reasoning

- Based on **models**

e.g. Equations, rules, constraints, formulas

$$(b_1 \wedge b_2 \wedge \neg b_3) \vee (\neg b_1 \wedge b_4 \wedge b_2)$$





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- Generic methods to perfom **complex tasks**

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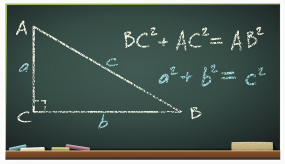

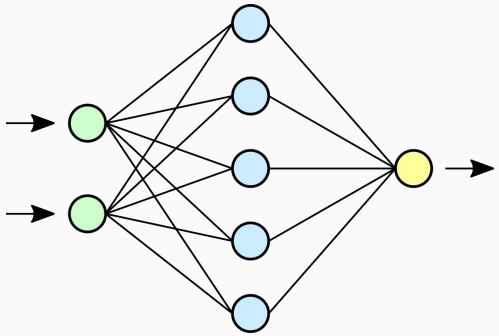

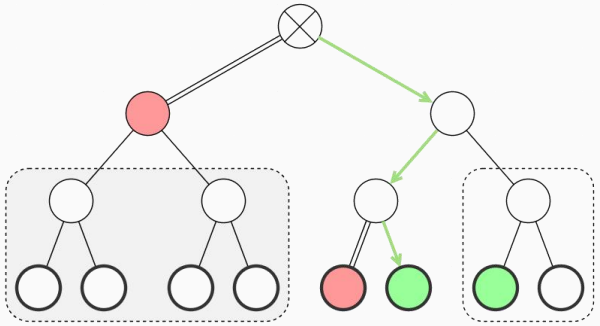

Automated learning

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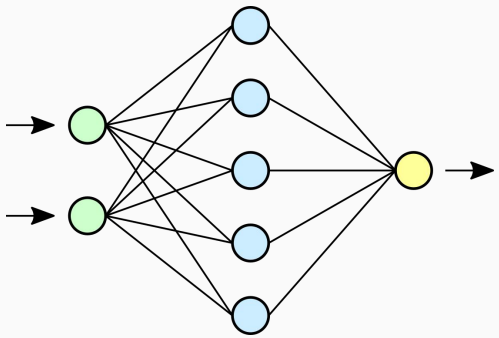

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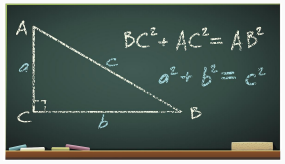
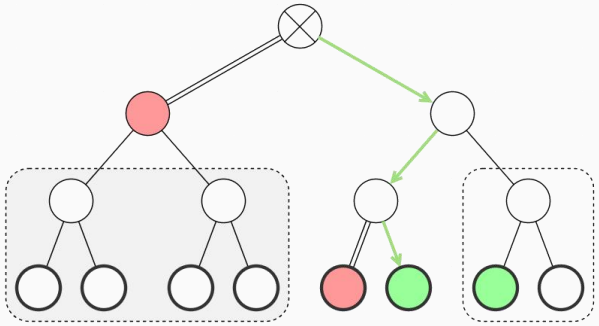


e.g. Guarantee that the flight control software computed optimal and safe trajectories



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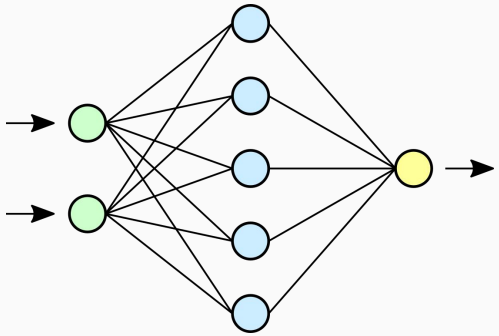

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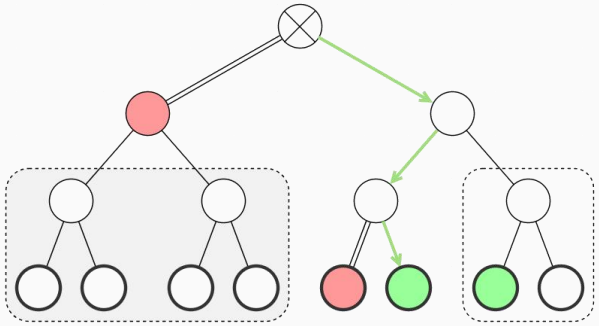

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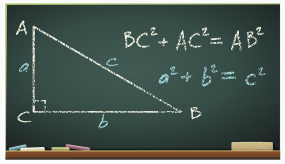

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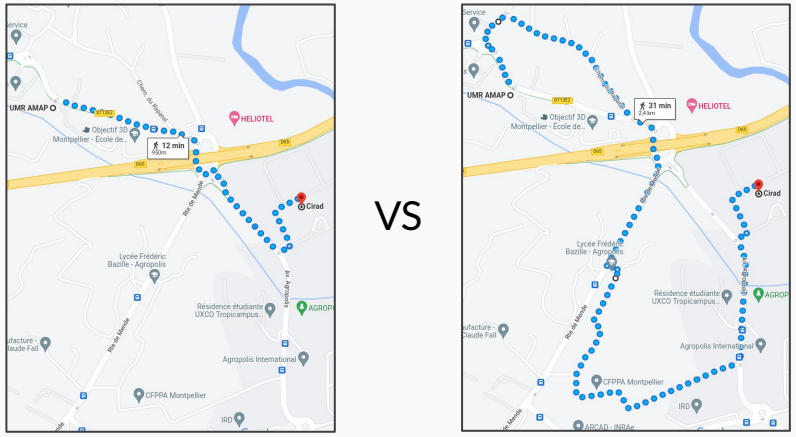
All roads lead to Rome, but some are faster than others...



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  « Constraint Programming represents one of the closest approaches computer has yet made to the Holy Grail of programming: ***the user states the problem, the computer solves it****. »*

Introduction to **Constraint programming (CP)**

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CSP: a triplet (X, D, C) where:

$X = \{x_1, \ldots, x_n\} \rightarrow$ variables
$D = \{D_1, \ldots, D_n\} \rightarrow$ domains
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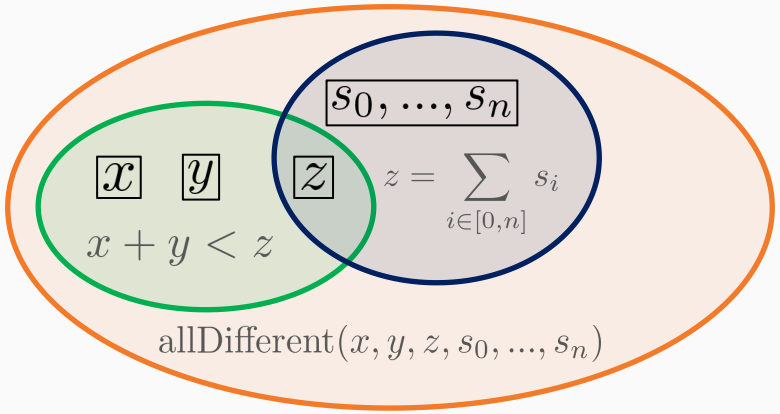

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- Variables can be of several types: *Integers*, *Reals*, *Sets*, *Graphs*, *etc.*

- Constraints can be **extensional**, **arithmetic**, and **logical** (semantic)

  possible values are enumerated

  e.g. AllDifferent, AtMost, Regular

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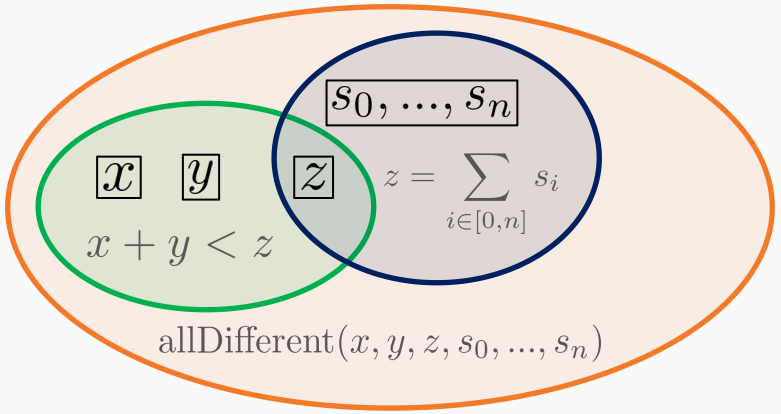

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- Solving is **generic**, and based on **Filtering**, **Propagation**, and **Backtracking**

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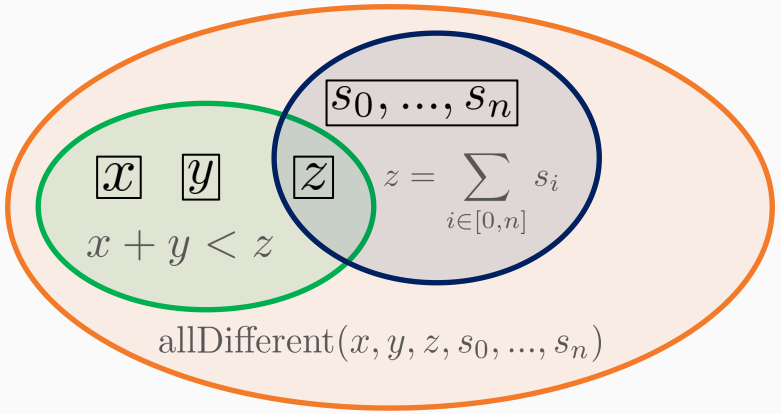


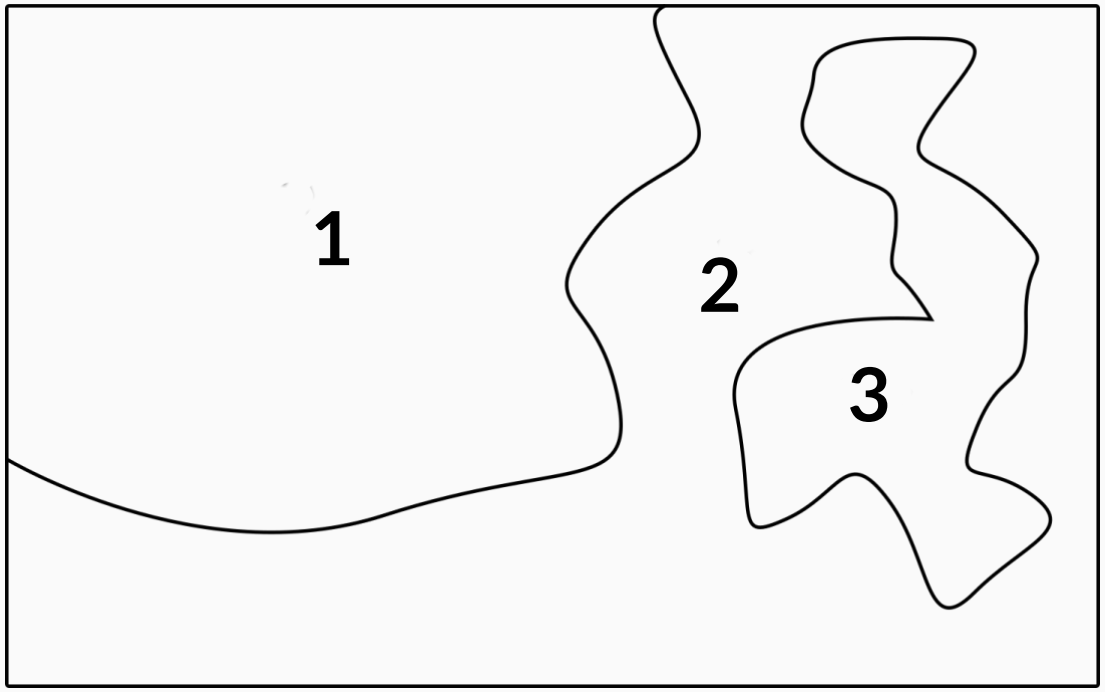
In summary, **CP** is:

- A generic approach for **modeling** and **solving constraint satisfaction** and **constrained optimization** problems
- Exact, expressive and extensible
- Support non-linear constraints
- A CP solver can interact with other systems
  - > e.g. Machine learning, SAT
- In practice:
  - > Modeling is ofter critical for performances
  - > When problems are hard, fine tuning is often necessary
- Many solvers, many languages

Google OR-Tools

Modelling in CP: a basic example

Fill each region with a different color
1 can be blue, green or red
2 and 3 can be blue or green



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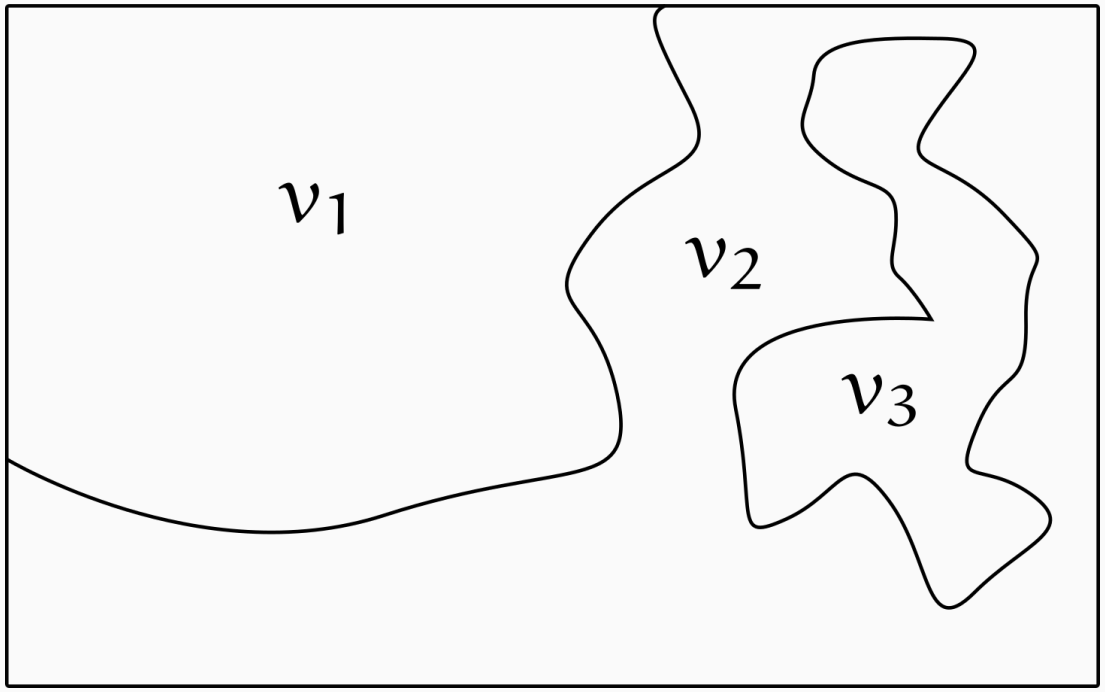

- For each region, one integer variable

$v_2 \in \{1, 2\}$

$v_1 \in \{1, 2, 3\}$

$v_3 \in \{1, 2\}$

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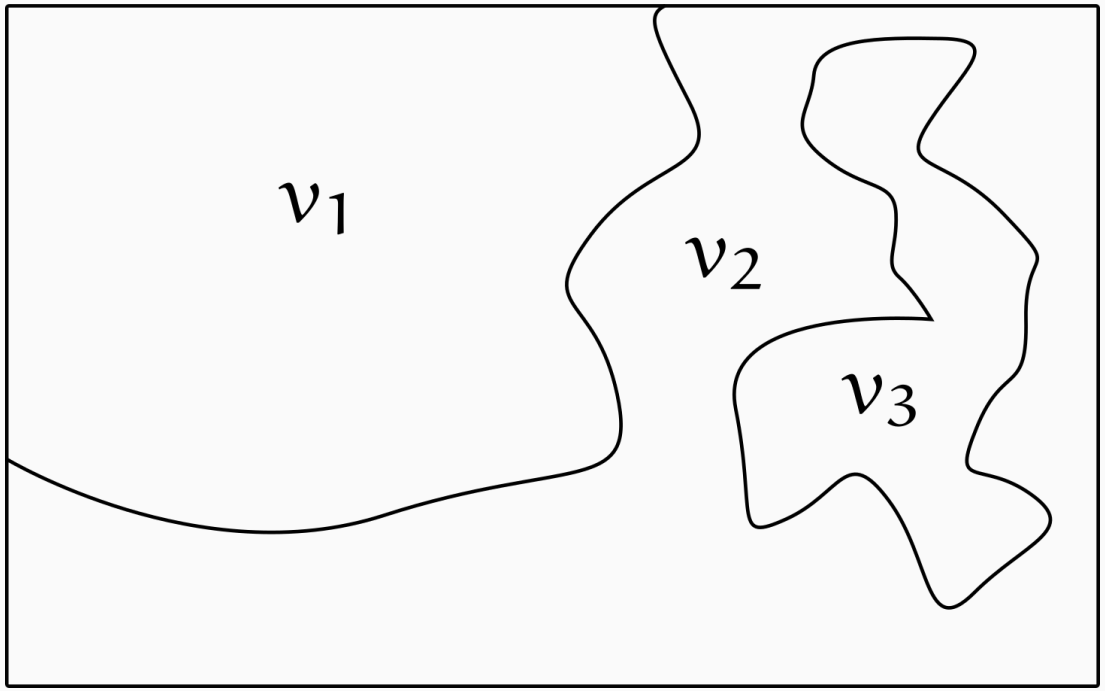
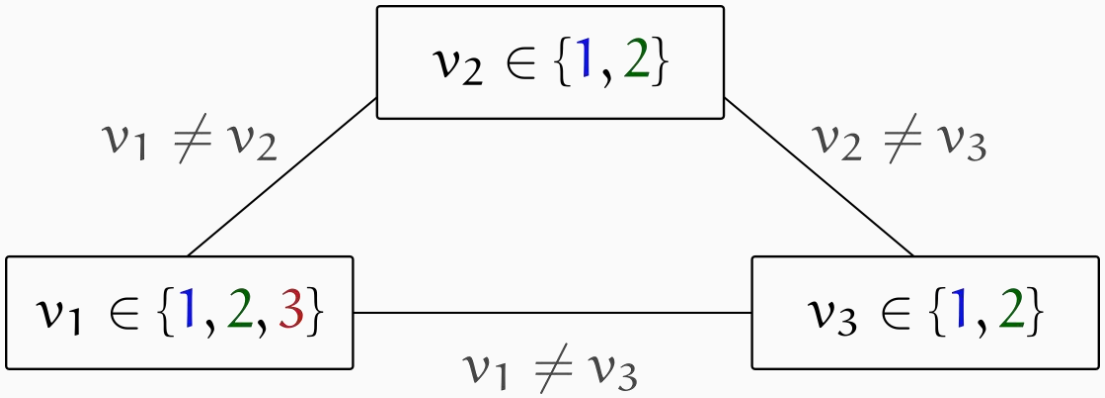


- For each region, one integer variable
- Inequality constraints



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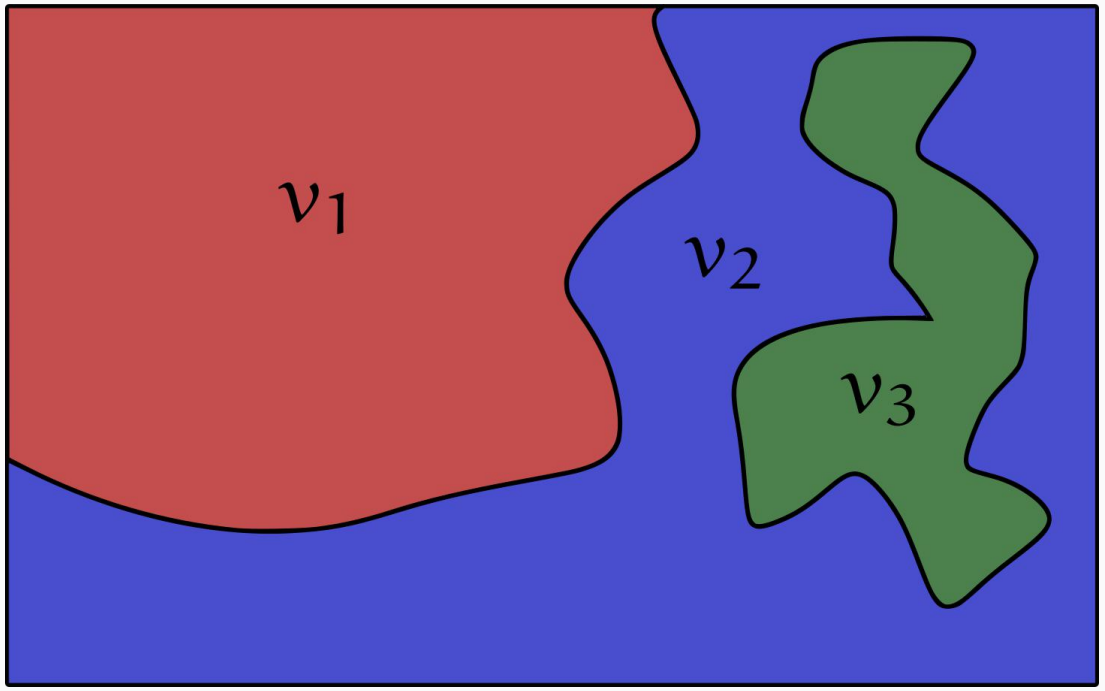
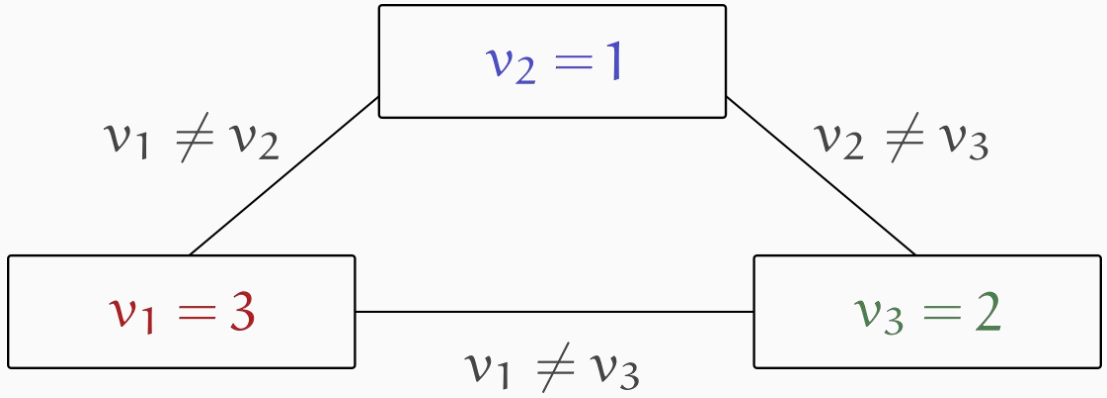
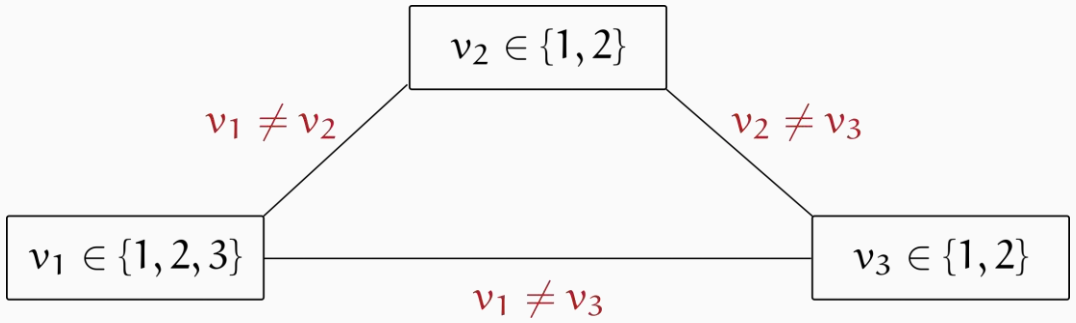


- For each region, one integer variable
- Inequality constraints
- An instantiation of the variables gives us a solution to the problem



What about the solving process ?



CP Model

Step	v_1	v_2	v_3	Deductions
#0 *Initialize*	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\emptyset$

Constraint propagation

$\mathcal{D}(v_1) = \{1,2,3\}$
$\mathcal{D}(v_2) = \{1,2\}$
$\mathcal{D}(v_3) = \{1,2\}$

Backtrackable search tree

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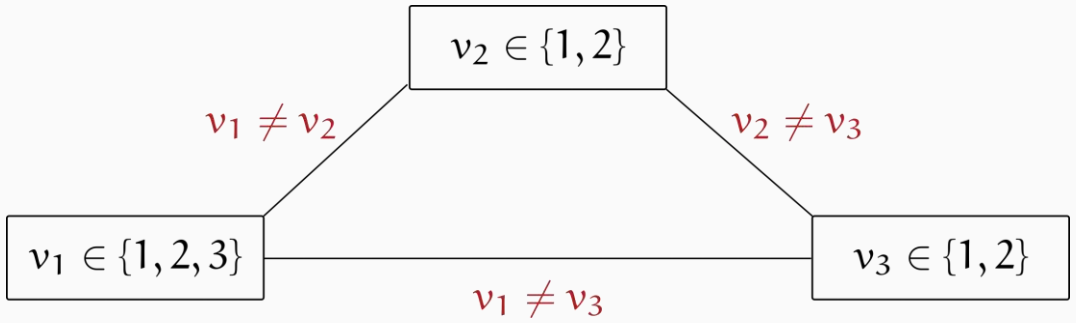

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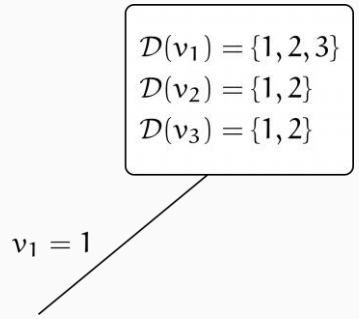
Constraint propagation



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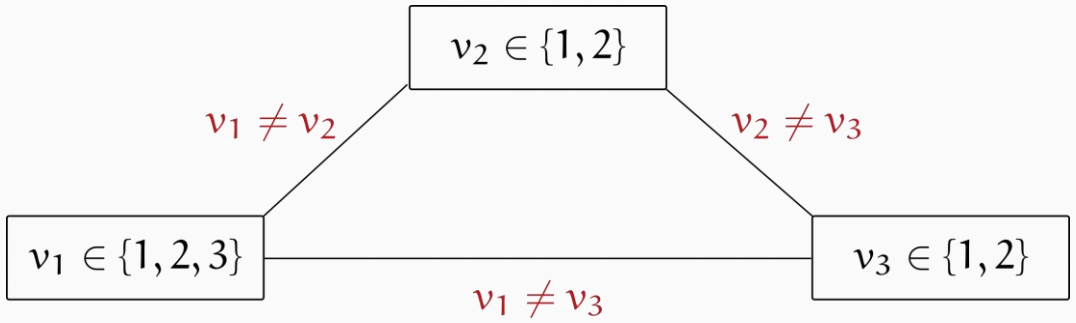

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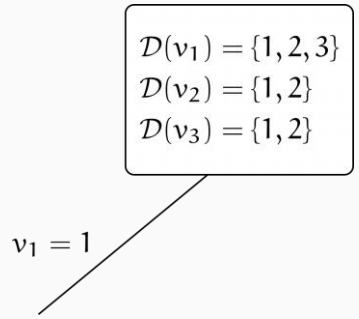

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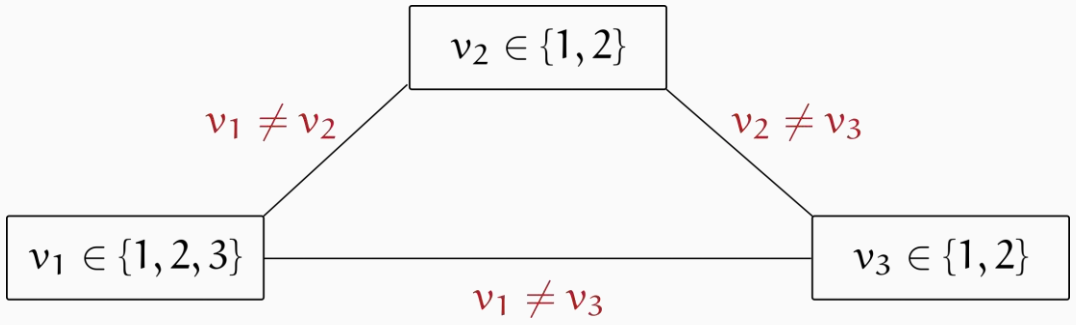

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Constraint propagation

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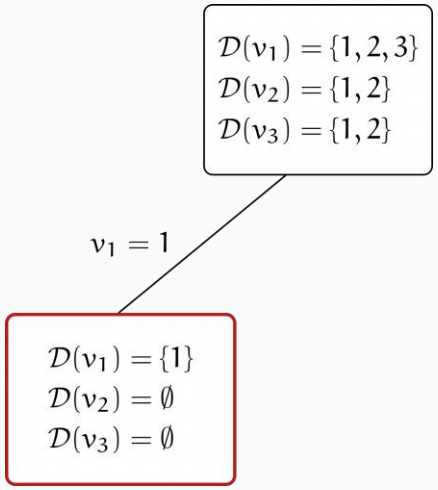

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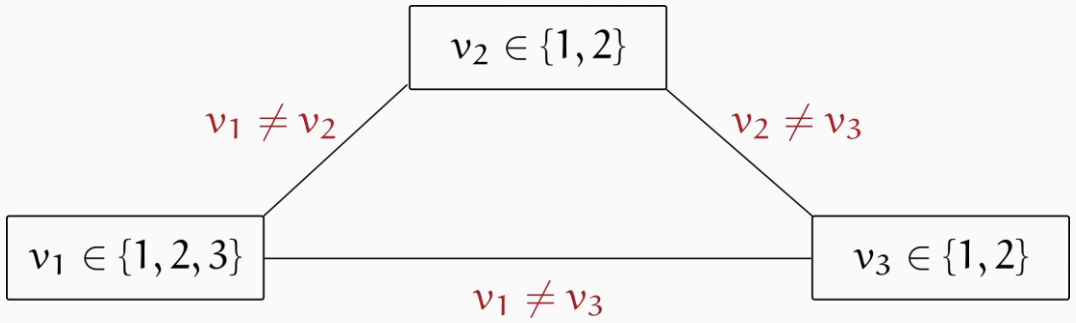

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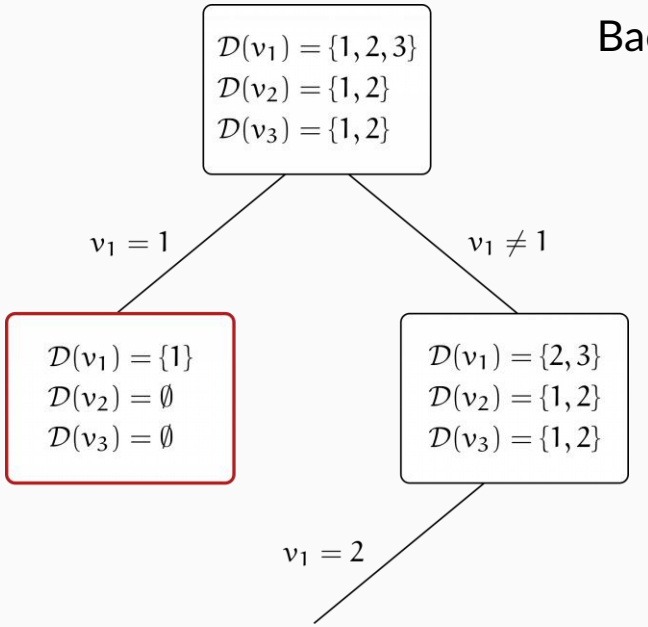
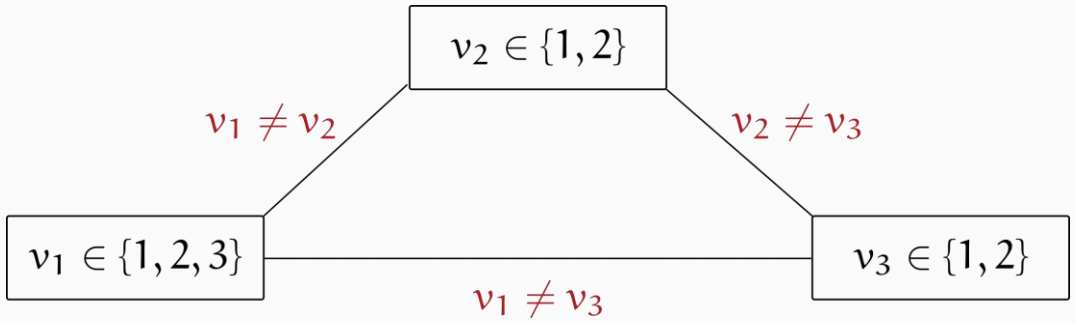
#3 *Remove* 2 *from* $\mathcal{D}(v_2)$ *and* $\mathcal{D}(v_3)$	–	$\emptyset$	$\emptyset$	$v_1 \neq 1$
#4 *Backtrack to* #0 *and try* $v_1 = 2$	$\{2\}$	$\{1,2\}$	$\{1,2\}$	$v_2 \neq 2 \wedge v_3 \neq 2$

Constraint propagation

Backtrackable search tree



What about the solving process ?



CP Model

Step	v_1	v_2	v_3	Deductions
#0 *Initialize*	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\emptyset$
#1 *Try* $v_1 = 1$	$\{1\}$	–	–	$v_2 \neq 1 \wedge v_3 \neq 1$
#2 *Remove* 1 *from* $\mathcal{D}(v_2)$ *and* $\mathcal{D}(v_3)$	–	$\{2\}$	$\{2\}$	$v_3 \neq 2 \wedge v_2 \neq 2$
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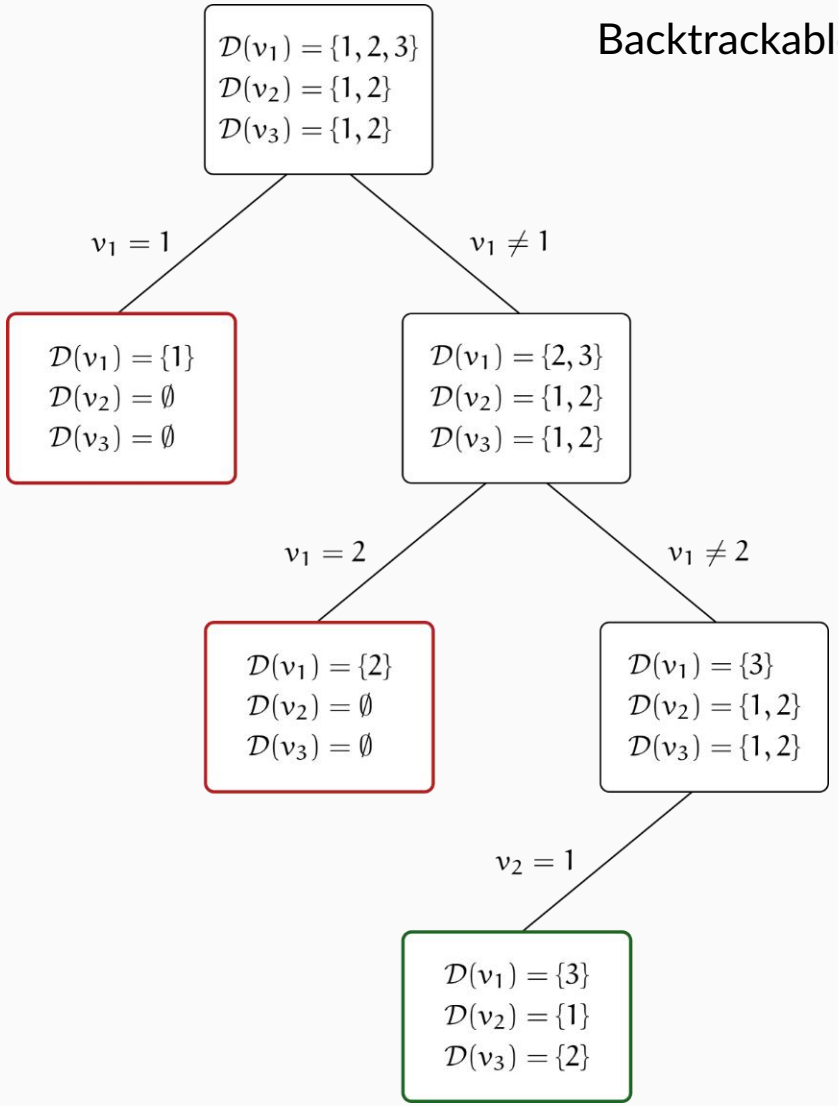

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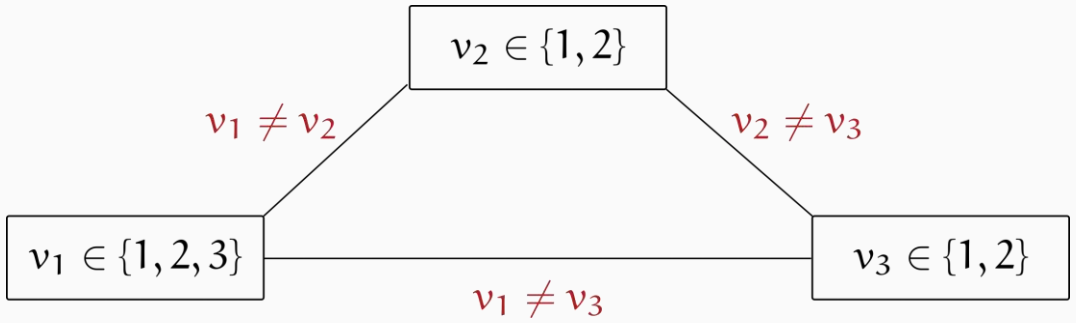

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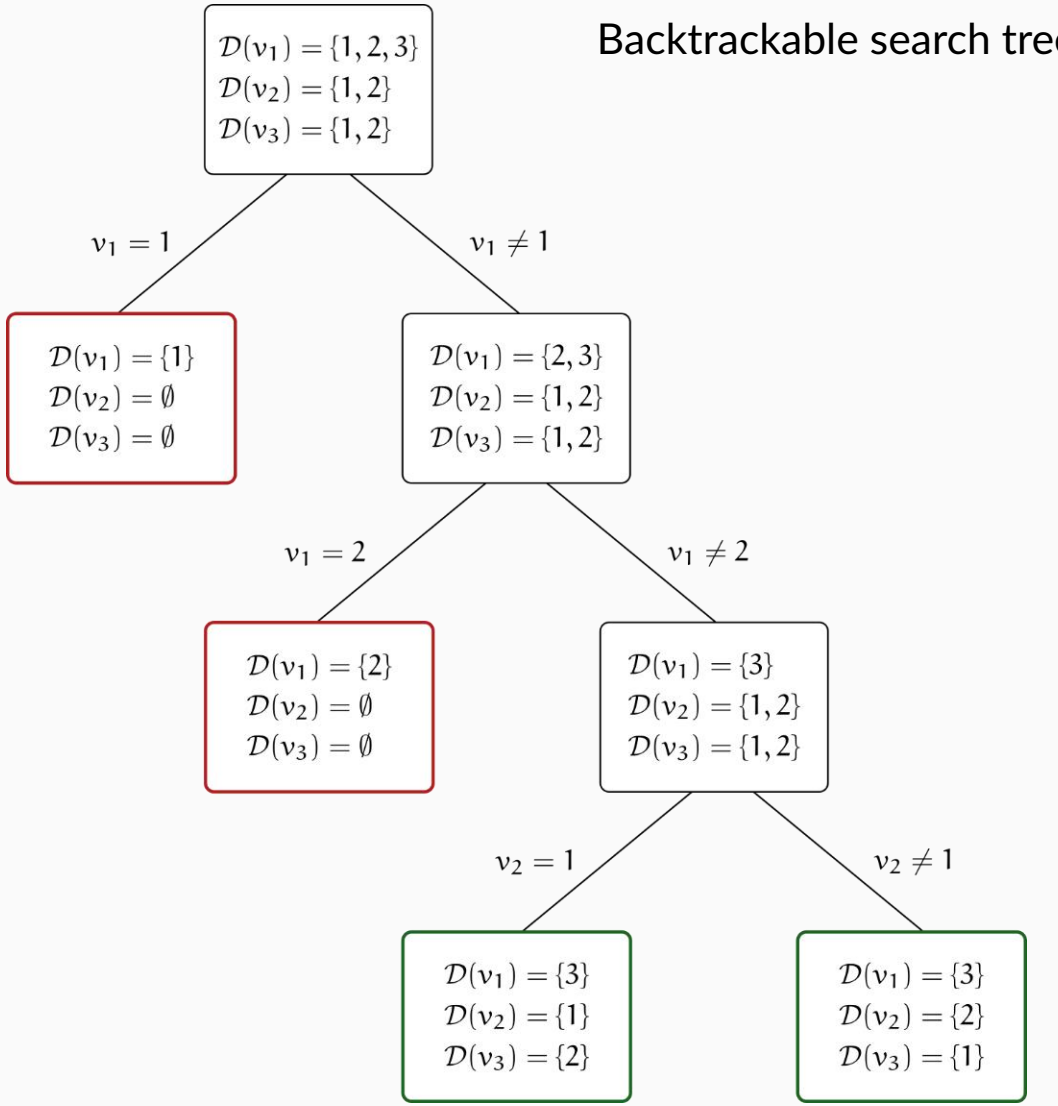
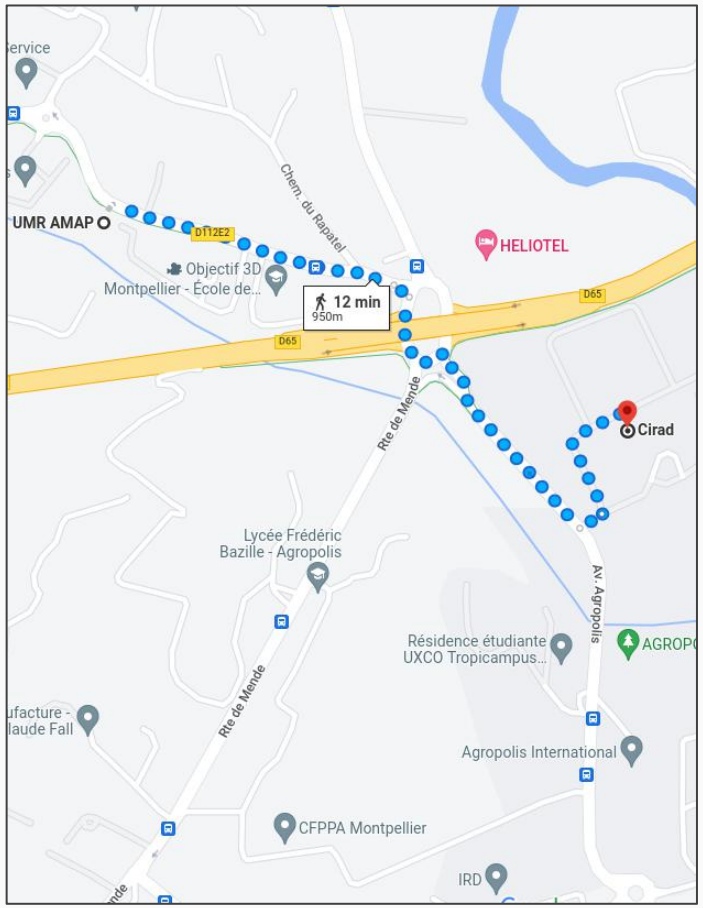
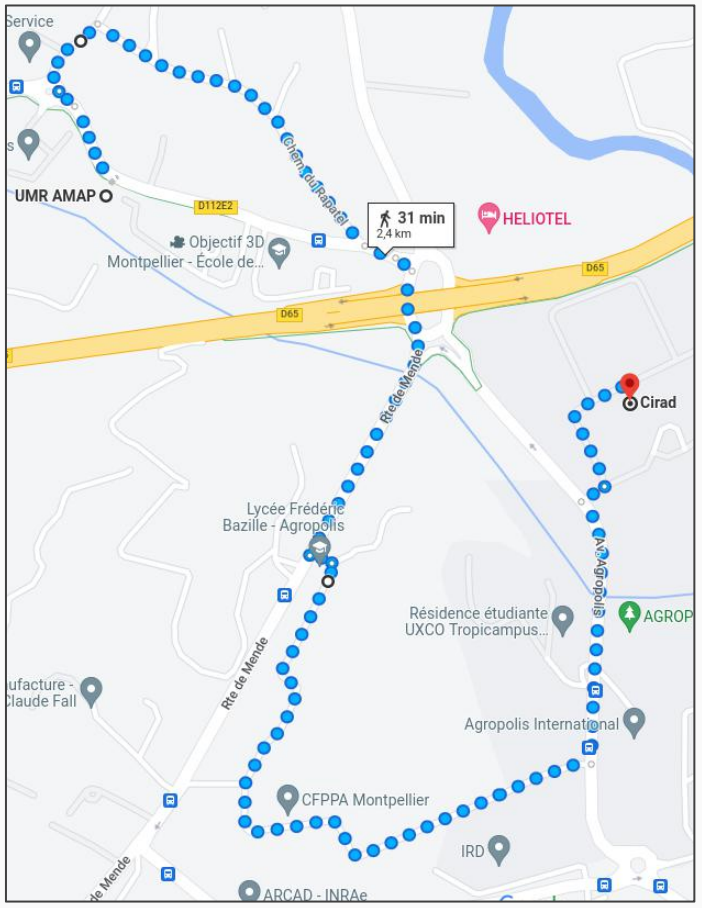


Do you remember ? *All roads lead to Rome, but some are faster than others...*

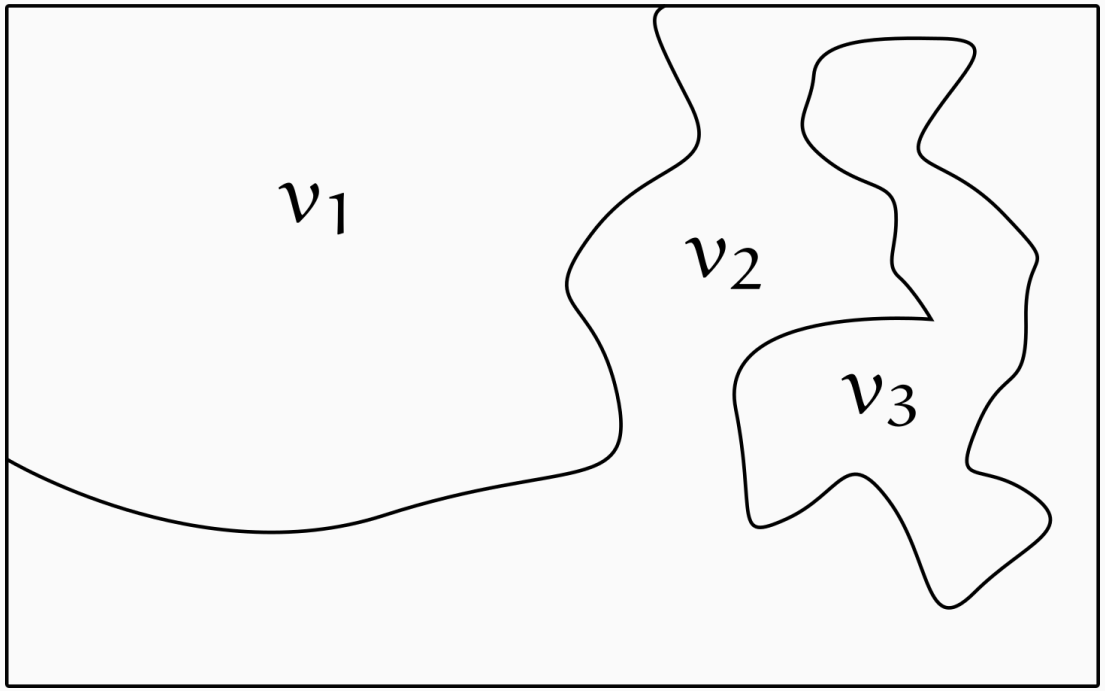


VS



A better model for our simple coloring problem ?

Fill each region with a different color
1 can be blue, green or red
2 and 3 can be blue or green



$v_2 \in \{1, 2\}$

$v_1 \in \{1, 2, 3\}$

$v_3 \in \{1, 2\}$

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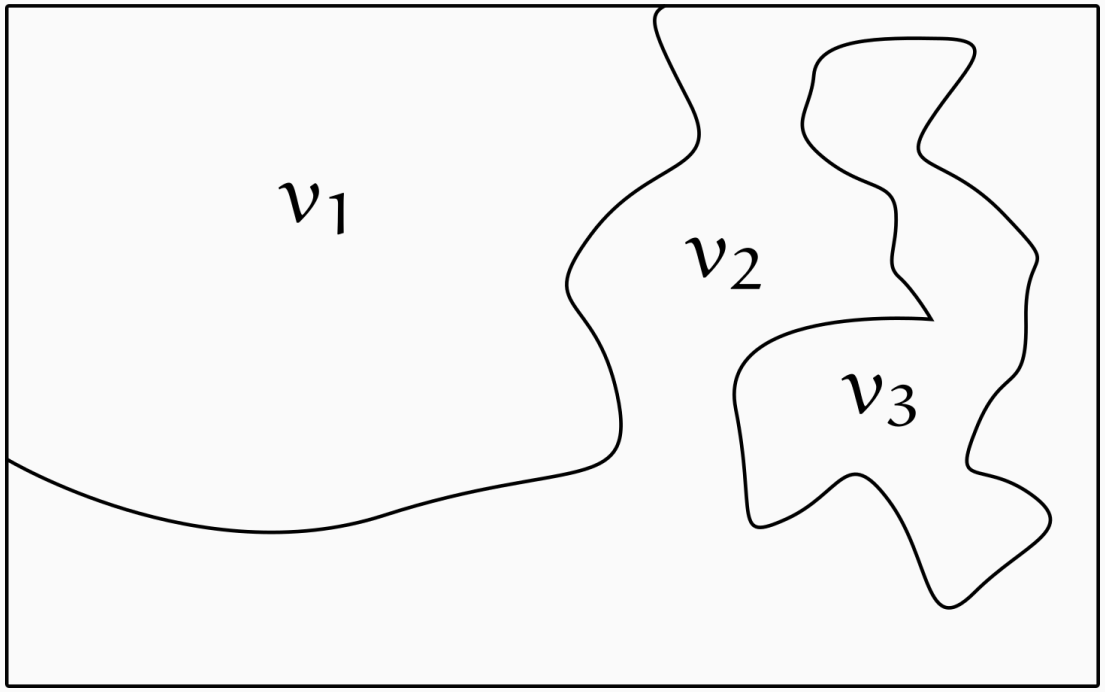


- **AllDifferent**, a global (logical) constraint

AllDifferent(v_1, v_2 , v_3)

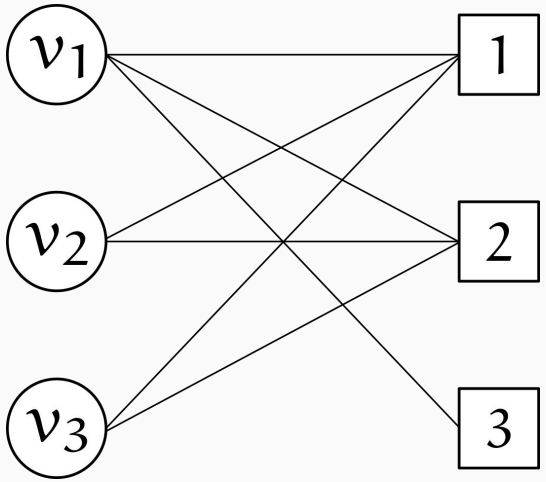
$v_2 \in \{1, 2\}$

$v_1 \in \{1, 2, 3\}$

$v_3 \in \{1, 2\}$

A global constraint example: **AllDifferent**

- Instead of applying pairwise binary inequality constraints,
- Variables and their possible values are seen as a **bipartite graph**



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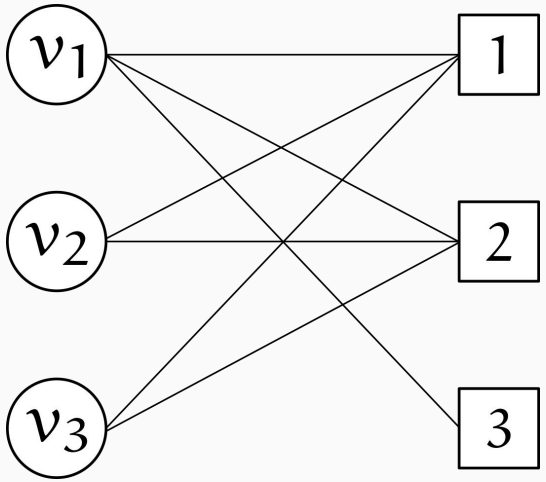
- A solution corresponds to a **maximum cardinality matching (MCM)**



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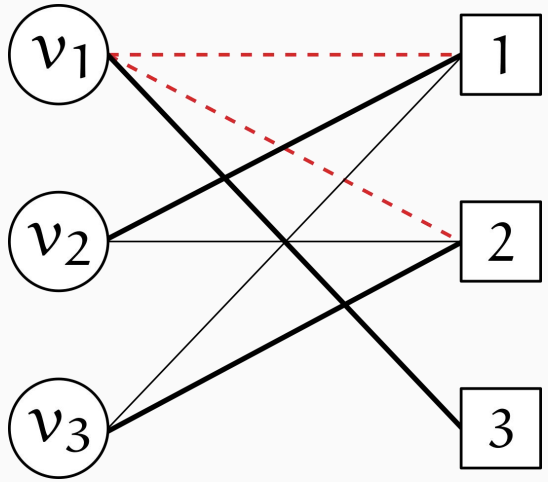
- Filtering: remove edges **that cannot belong** to a MCM
  - > Algorithm based on the Hall's marriage theorem (linear time)



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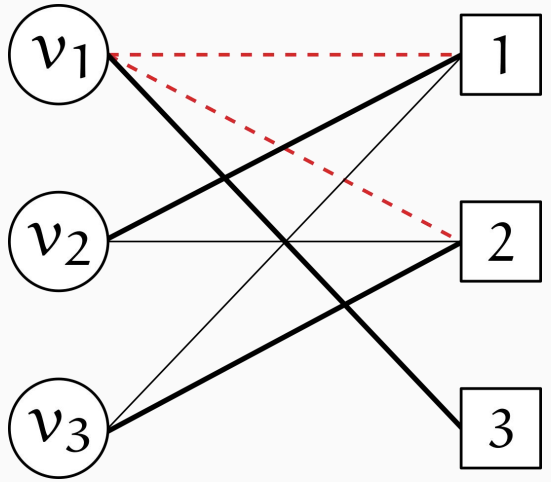

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⇨

Step	v_1	v_2	v_3	Deductions
#0 *Initialize*	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$v_1 \neq 1 \wedge v_1 \neq 2$
#1 *Remove 1 and 2 from* $\mathcal{D}(v_1)$	$\{3\}$	–	–	$\emptyset$
#2 *Try* $v_2 = 1$	–	$\{1\}$	–	$v_3 \neq 1$
#3 *Remove 1 from* $\mathcal{D}(v_3)$	$\{3\}$	$\{1\}$	$\{2\}$	$(3,1,2)$ is a solution

How to solve a Sudoku with CP ?

	1	2		3	4	5	6	7
	3	4	5		6	1	8	2
		1		5	8	2		6
		8	6					1
	2				7		5	
		3	7		5		2	8
	8			6		7		
2		7		8	3	6	1	5

How to solve a Sudoku with CP ?

	1	2		3	4	5	6	7
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	8			6		7		
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For each cell i, an integer variable:

- $x_i \in [1, 9]$ *if empty*
- $x_i = c$ *if not empty*

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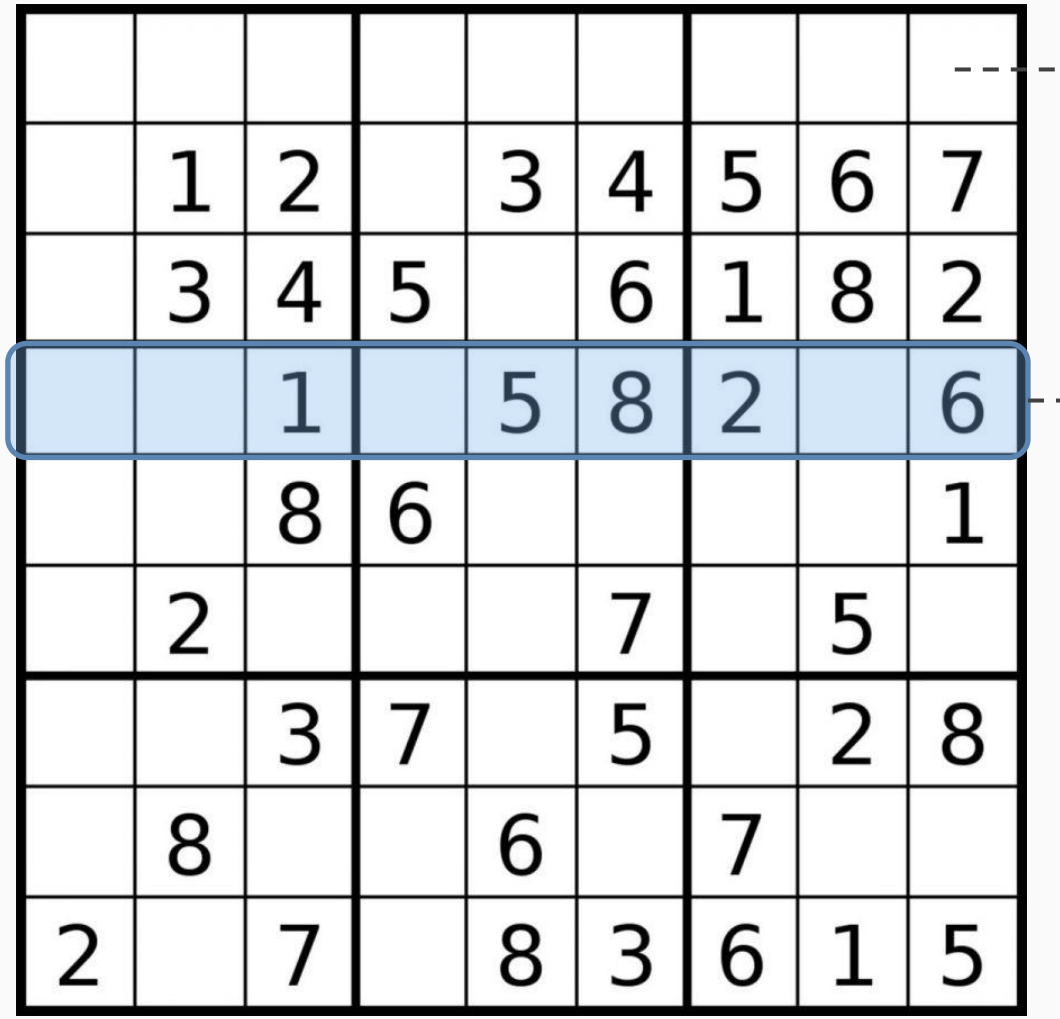

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For each set of variables R_i *in a row:*

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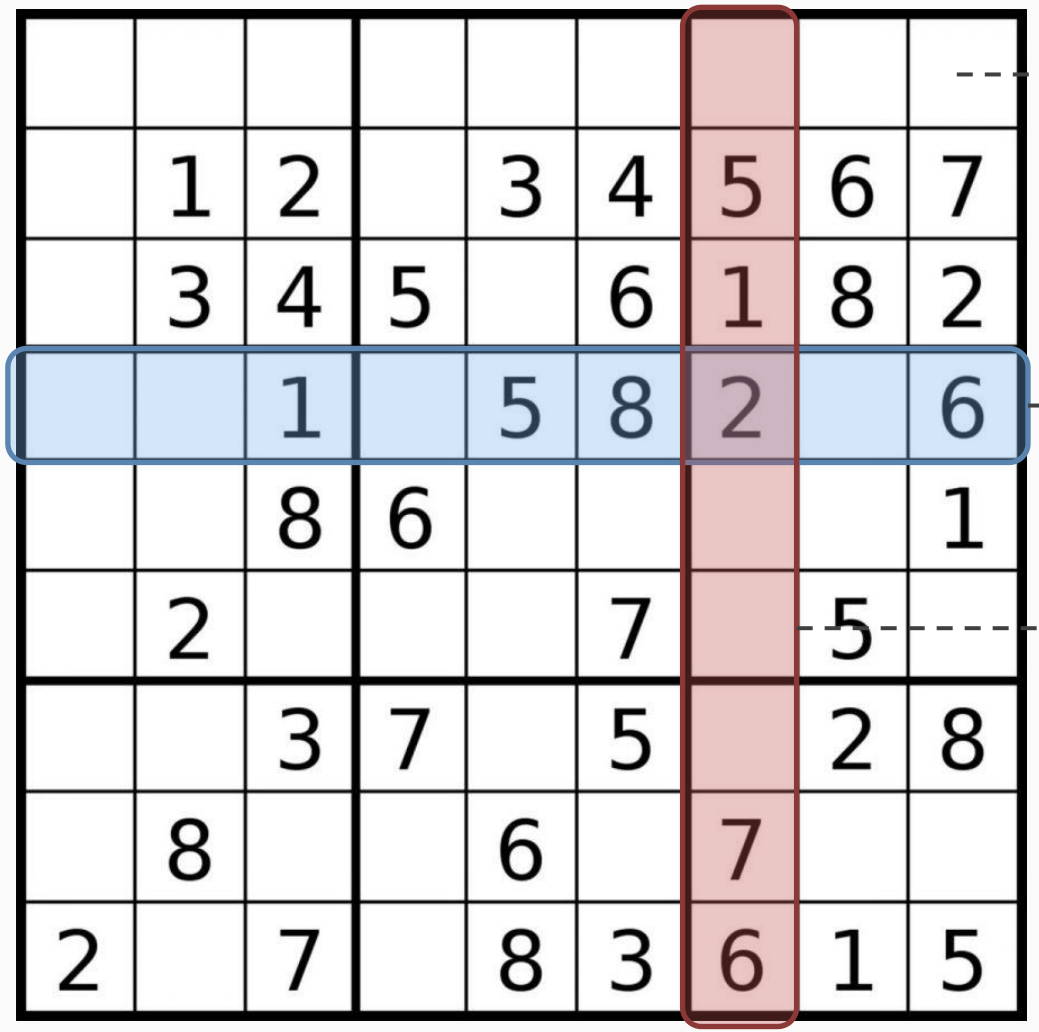

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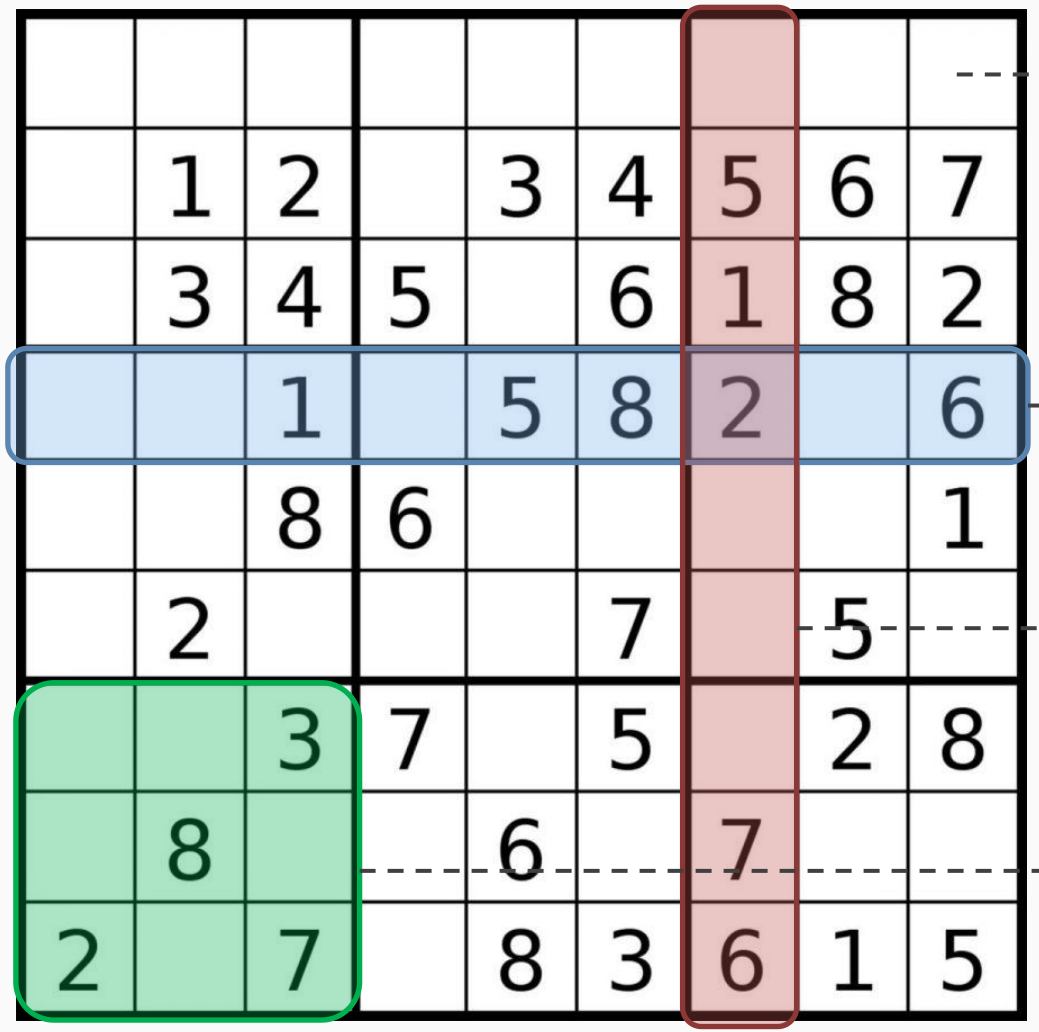
How to solve a Sudoku with CP ?



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For each set of variables C_i *in a column:*

- *AllDifferent*(C_i)

For each set of variables S_i *in a square:*

- *AllDifferent*(S_i)

Thank you !


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